Chapter 10

Application of GKSL Master Equation of Infinitesimal Generator of Decoherence-free Subalgebra of Uniformly Continuous Quantum Markov Semigroup

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ABSTRACT

In this work, we develop an analog of Gorini-Kossakowski-Sudarshan-Lindblad(GKSL) for the infinitesimal generator of decoherence-free subalgebra of Quantum Markov semigroup. The GKSL developed was then applied on $M_n(\mathbf{c})$. The result showed that the matrix space $M_n(\mathbf{c}) = V_0 \oplus V_-$ will not decohere under N(T), i.e V_0 = N(T) implies $M_n(\mathbf{c})$ = N(T) since $V- = \{0\}$. Also by introducing GKSL of N(T) on $M_n(\mathbf{c})$, $L(x^*x) = 0$.
Keywords: Decoherence-free subalgebra, Infinitesimal generator, quantum Markov semigroup, eigenvalue and eigenvector.

INTRODUCTION

A subspace of a quantum system Hilbert space that is invariant to non-unitary dynamics is called a decoherence-free subspace (DFs). The subject of quantum information processing (QIP) began the study of decoherence-free subspace with a search for structured methods to avoid decoherence. Identification of particular states which have the potential of being unchanged by certain decohering processes (i.e certain interaction with the environment)was included as one of the methods. These studies started with observations made by Pazy, 1983 who studied the consequences of pure dephasing on two qubits that have the same interaction with the environment. They found that two such qubits do not decohere. They also characterized decoherence- free subspace (DFs) as a special class of quantum error correcting codes.

In the approach of Blanchard and Olkiewicz, 2006, if the space is an algebraic space (i.e a von Neumann or a C*-algebra), the decoherence-free subspace becomes a decoherence-free subalgebra. Therefore, if a von Neumann algebra is a quantum system (say open), there is a part of the algebra that behaves like a closed system (i.e a part that is decoherence-free). This can be used to study the asymptotic behaviour of the quantum system.The evolution of a closed quantum system which does not interact with the environment, can be described by a one-parameter group of automorphism (at)t≥0 , with at(x) = e itHxe−itH and H self-adjoint. Inside an open Quantum system, sometimes, one can have a subsystem evolving like a closed quantum system where the typical effects of the interaction with the environment do not appear and the typical quantum features of the system, like quantum coherence and entanglement of quantum states are preserved (Dhari *et al.*, 2010). Some years ago, alots of people have been showing interest in the use of Quantum Markov Semigroup to model open quantum systems having subsystems which are not affected by decoherence. In these applications, the Quantum Markov Semigroup (in the Heisenberg picture) acts as a semigroup of automorphisms of a von Neumann subalgebra N(T) of B(h), called the Decoherence-free subalgebra. This subalgebra allows identification of noise protected subsystems where states evolve unitarily, moreover its structure and relationship with the set of fixed points also has important consequences on the asymptotic behavior of the Quantum Markov Semigroup. Decoherence-free subalgebra allows us to gain insight into the structure of a Quantum Markov Semigroup, its invariant states and environment induced decoherence. Indeed, its structure as a von Neumann algebra has important consequences on the structure and the action of the whole Quantum Markov Semigroup (Fagnola& Rebolledo, 2008).

This paper focuses on the application of Gorini-Kossakowski-Sudarshan-Lindblad(GKSL) master equation of infinitesimal generator of decoherence-free subalgebra of uniformly continuous Quantum Markov semigroup.The GKSL was applied on the matrix space Mn(c) = V0 ⊕ V−.

The behaviours of the matrix space under Decoherence-free Subalgebra of Quantum Markov Semigroup and GKSL master equation were the major concern of this paper.

THE DECOHERENCE-FREE SUBALGEBRA OF A QUANTUM MARKOV SEMIGROUP(QMS)

Suppose h is a complex separable Hilbert space. A QMS B(h) is a semigroup T = (Tt)t≥0 of completely positive, identity preserving normal maps which is also weakly-∗-continuous. If T is uniformly continuous, namely norm continuous, its generator L can be represented in the well-known (Ogundiran, 2015; Sinha & Goswami, 2007) Gorini-KossakowskiSudarshan-Lindblad (GKSL) form as

$$\mathbf{L}(x) = i[H, x] - \frac{1}{2}\sum_{l\geq 1}(L_l^* L_l x - 2L_l^* x L_l + x L_l^* L_l)$$

where $H = H^*$ and $(L_l)_{l\geq 1}$are operators on h such that the series $\sum_{l\geq 1} L_i{}^* L_l$ is strongly convergent and [.,.] denotes the commutator $[x, y] = xy - yx$. The decoherence-free (DF) subalgebra of T is defined by;

$$N(T) = \{x \in B(h) : T_t(x^* x) = T_t(x)^* T_t(x), T_t(x x^*) = T_t(x) T_t(x)^* \ \forall\, t \geq 0\}.$$

It is a well-known fact that N(T) is the biggest von Neumann subalgebra of B(h) on which every Tt acts as a ∗-endomorphism (Evans,1977).

Preliminaries

The following definitions, lemmas and propositions are necessary for our results

Definition 1 (Bratelli *et al.*, 1987): A normed vector space is vector space (V, F) associated with a function $\|\cdot\|: v \to \mathbb{R}$, called a norm. that obeys the following axioms

(1) ∀ u∈V,‖u‖≥0
(2) ∀ v∈V,∀α∈F,‖αv‖=|α|‖v‖
(3) ∀ v,u∈V,‖u+v‖≤‖u‖+‖v‖

Definition 2 (Bratelli *et al.*, 1987): Let V be a vector space. An inner product on V is a rule that assigns to each pair $v, w \in V$ a real number $\langle v, w \rangle$ such that, for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$,

(i) $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0$,
(ii) $\langle v, w \rangle = \langle w, v \rangle$,
(iii) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,
(iv) $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$

Definition 3 (Bratelli *et al.*, 1987): Suppose X be a nonempty set. A function P: X × X → R is known as a metric provided for all x, y, and z in X,
(1) P(x, y) > 0,
(2) P(x, y) = 0 if and only if x = y,
(3) P(x, y) = P(y, x),
(4) P(x, y) ≤ P(x, z) + P(z, y)
A metric space is made up of a nonempty set and a metric on the set.
Definition 4 (Bratelli *et al.*, 1987): Let A be an n × n matrix and let X ∈ C n be a nonzero vector for which AX = λX for some scalar λ. Then λ is called an eigenvalue of the matrix A and X is called an eigenvector of A associated with λ, or a X-eigenvector of A.
Definition 5 (Parthasarathy, 1992): The infinitesimal generator of the Quantum Markov semigroup {Tt, t ≥ 0} is a linear (but not necessarily bounded) operator L : D(L) → A

$$\mathbf{L}(a) = \lim_{t \to 0} \frac{T_t(a) - a}{t}, a \in D(\mathbf{L}).$$

Where D(L) is the domain of L.
Definition 6 (Bratelli *et al.*, 1987): Suppose U is a vector space over a field C, U is called an algebra if it is equipped with a multiplication law which associates to each pair A, B ∈ U, the product A · B and for every A, B, C ∈ U, the following properties are satisfied
(1) A.(B.C) = (A.B).C (associativity),
(2) A.(B + C) = A.B + A.C (distributive); and
(3) αβ(A.B) = (αA).(βB); α, β ∈ C.
Definition 7 (Bratelli *et al.*, 1987): A subspace B of U which is also an algebra with respect to the operations of U is called a subalgebra.

Definition 8 (Bratelli *et al.*, 1987): A mapping $A \in U \rightarrow A* \in U$ is called an involution or adjoint operation of the algebra U if it has the following properties:

(1) $A** = A$,

(2) $(AB)* = B*A*$; and

(3) $(\alpha A + \beta B)* = \alpha A* + \beta B*$,

where α, β are complex conjugates of α and β respectively.

Definition 9 (Bratelli *et al.*, 1987): An algebra with an involution is called a ∗-algebra.

Definition 10 (Parthasarathy, 1992): A von Neumann algebra on H is a ∗ -subalgebra A of $L^{\infty}(H)$ such that $A = A^{II}$.

Definition 11 (Accardi *et al.*, 1999): A quantum dynamical semigroup (QDS) on Von Neumann algebra A is a family of bounded linear maps {Tt, t ≥ 0} on A that satisfies the following conditions:

(1) $T_0(a) = a$, for all a ∈ A;

(2) $T_{t+s}(a) = Tt(Ts(a)) = Ts(Tt(a))$, for all s, t ≥ 0 and all a ∈ A;

(3) Tt is completely positive for all t ≥ 0;

(4) Tt is a normal operator on A for all t ≥ 0, that is, for every increasing net (aα)α in A with Tt(vαaα) = vαTt(aα) whenever vαaα ∈ A; or equivalently, Tt is a σ-weakly continuous operator in A for all t ≥ 0. That is, the map a → tr(ρTt(a)) is continuous from A to C for each ρ ∈ S(A), where S(A) is a set of quantum state on A.

Definition 12 (Agredo, 2014): A quantum dynamical semigroup {Tt, t ≥ 0} on A is said to be a quantum Markov (respectively, sub-Markov) semigroup if

Tt(I) = I, (respectively, Tt(I) ≤ I) ∀ t ≥ 0

MAIN RESULTS

Here, we develop an analog of (GKSL) for the infinitesimal generator of decoherence-free subalgebra of Quantum Markov semigroup.

Definition 13: The infinitesimal generator of the Quantum Markov semigroup {Tt, t ≥ 0} on a decoherence-free subalgebra will be defined as a linear (but not necessarily bounded) operator L:D(L) → N(T) such that

$$\mathbf{L}(x^*x) = \lim_{t\to 0} \frac{T_t(x^*x) - x^*x}{t}, \; x^*x \in D(\mathbf{L}).$$

The domain of L, denoted by D(L), is the collection of x ∗x ∈ N(T) such that the above limit exists.

Theorem:

The GKSL equation

$$\mathbf{L}(x) = \frac{-1}{2}\left(\sum_{l\geq 1} L_l^* L_l x + \sum_{l\geq 1} x L_l^* L_l - 2\sum_{l\geq 1} L_l^* x L_l\right) + i[H,x] \; \forall \; x \in N(T)$$

in terms of the decoherence-free subalgebra of a Quantum Markov semigroup is

$$\mathbf{L}(x^*x) = -\left(\sum_{l\geq 1} L_l^* L_l x^* x + \sum_{l\geq 1} x^* x L_l^* L_l - 2\sum_{l\geq 1} L_l^* x^* x L_l\right) + 2i[H, x^*x].$$

Proof:

Suppose the infinitesimal generator of a Quantum Markov semigroup on X is

$$\mathbf{L}(x) = \lim_{t\to 0} \frac{T_t(x) - x}{t}, \quad x \in D(L).$$

where the domain of L, denoted by D(L), is the collection of x ∈ X for which the limit exists.

Let x ∈ N(T). Then

$$\begin{aligned}\mathbf{L}(x^*x) &= \lim_{t\to0}\frac{T_tx^*x - x^*x}{t}\\ &= \lim_{t\to0}\frac{T_tx^*T_tx - x^*x}{t}\\ &= \lim_{t\to0}\frac{(T_t(x^*)-x^*)}{t}T_t(x) + \lim_{t\to0}x^*\frac{(T_t(x)-x)}{t}\\ &= \lim_{t\to0}\frac{T_t(x^*)-x^*}{t}\lim_{t\to0}T_t(x) + x^*\lim_{t\to0}\frac{T_t(x)-x}{t}\\ &= \lim_{t\to0}\frac{T_t(x^*)-x^*}{t}x + x^*\lim_{t\to0}\frac{T_t(x)-x}{t}\\ &= x^*\lim_{t\to0}\frac{T_t(x)-x}{t} + \lim_{t\to0}\frac{T_t(x^*)-x^*}{t}x\end{aligned}$$

$$\mathbf{L}(x^*x) = x^*\mathbf{L}(x) + \mathbf{L}(x^*)x.$$

The GKSL form of L is

$$\mathbf{L}(x) = \frac{-1}{2}\left(\sum_{l\geq1}L_l^*L_lx + \sum_{l\geq1}xL_l^*L_l - 2\sum_{l\geq1}L_l^*xL_l\right) + i[H,x].$$

Substituting the GKSL equation in (1), we have

$$\mathbf{L}(x^*x) = x^*\left(\frac{-1}{2}\left(\sum_{l\geq1}L_l^*L_lx + \sum_{l\geq1}xL_l^*L_l - 2\sum_{l\geq1}L_l^*xL_l\right)\right.$$

$$\left.+i[H,x] + \left(\frac{-1}{2}\left(\sum_{l\geq1}L_l^*L_lx^* + \sum_{l\geq1}x^*L_l^*L_l - 2\sum_{l\geq1}L_l^*x^*L_l\right) + \right.\right.$$

$$i[H,x^*]x = \frac{-1}{2}\left(\sum_{l\geq1}L_l^*L_lx^*x + \sum_{l\geq1}x^*xL_l^*L_l - 2\sum_{l\geq1}L_l^*x^*xL_l\right)$$

$$+ix^*[H,x]\frac{-1}{2}\left(\sum_{l\geq1}L_l^*L_lx^*x + \sum_{l\geq1}x^*xL_l^*L_l - 2\sum_{l\geq1}L_l^*x^*xL_l\right) + i[H,x^*]x$$

$$= \frac{-1}{2}\left(\sum_{l\geq1}L_l^*L_lx^*x + \sum_{l\geq1}L_l^*L_lx^*x + \sum_{l\geq1}x^*xL_l^*L_l + \sum_{l\geq1}x^*xL_l^*L_l - 2\sum_{l\geq1}L_l^*x^*xL_l - 2\sum_{l\geq1}L_l^*x^*xL_l)\right.$$

$$+ix^*[H,x] + i[H,x^*]x = \frac{-1}{2}\left(2\sum_{l\geq1}L_l^*L_lx^*x + 2\sum_{l\geq1}x^*xL_l^*L_l - 4\sum_{l\geq1}L_l^*x^*xL_l\right)$$

$$+2i[H,x^*x] = -\left(\sum_{l\geq1}L_l^*L_lx^*x + \sum_{l\geq1}x^*xL_l^*L_l - 2\sum_{l\geq1}L_l^*x^*xL_l\right) + 2i[H,x^*x].$$

Hence, the GKSL equation in terms of the decoherence-free subalgebra of a Quantum Markov semigroup is

$$\mathbf{L}(x^*x) = -\left(\sum_{l\geq 1} L_l^* L_l x^* x + \sum_{l\geq 1} x^* x L_l^* L_l - 2\sum_{l\geq 1} L_l^* x^* x L_l\right) + 2i[H, x^*x]. \quad (2)$$

Example 1: Let

$$V_0 := Span\{x \in M_n(c) : L(x) = i\lambda x \text{ for some } \lambda \in R\}$$

$$V_- := \oplus Re_{\lambda<0} N_\lambda^L$$

$$M_n(c) = V_0 \oplus V_-$$

$$N(T) = \{x \in B(h) : T_t(x^*x) = T_t(x)^* T_t(x), T_t(xx^*) = T_t(x) T_t(x)^* \,\forall\, t \geq 0\}.$$

Consider a QMS T generated by L(x) = i[H, x]. Then we have V0 = N(T) = Mn(c), simply because the semigroup is

$$\begin{aligned} T_t(x^*x) = T_t(x^*)T_t(x) &= e^{itH}x^*e^{-itH}.e^{itH}xe^{-tH} \\ &= e^{itH}x^*xe^{-itH} \\ &= e^{itH}x^*xe^{-itH} \;\forall\; x, \end{aligned}$$

so N(T) = Mn(c), V− = {0} and consequently V0 = N(T).

Example 2:

Let u, v in c n be two linearly independent vectors where k u k=k v k= 1. Introducing a generator L on Mn(c) by

$$-\mathbf{L}(x^*x) = |v\rangle\langle u,u\rangle\langle(x^*x)^*v| - 2|v\rangle\langle u|, x^*xu\rangle\langle v| + |x^*xv\rangle\langle u,u\rangle\langle v|.$$

Since $\langle u, u\rangle = \| u \|^2 = 1$ we have

$$-\mathbf{L}(x^*x) = |v\rangle\langle(x^*x)^*v| - 2\langle u, x^*xu\rangle|v\rangle\langle v| + |x^*xv\rangle\langle v|$$

$$-\mathbf{L}(x^*x) = |v\rangle\langle xx^*v| - 2\langle u, x^*xu\rangle|v\rangle\langle v| + |x^*xv\rangle\langle v|.$$

This is the generator of a QMS since it can be written in the lindblad form with H = 0 and $= L_1 = |u\rangle\langle v|$, L_1 for $l \geq 2$. Moreover, we have N(T) = {L, L∗} 0 ⊆ Ker(L) ⊆ V_0.We want to prove that, in some cases, V_0 is strictly bigger than N(T) and is not always an algebra. We characterize V0, KerI and N(T) and determine the invariant states for these generators. In order to do this, it will be useful to know the spectrum of L. So consider an eigenvalue λ of the generator and notice that

$$\mathbf{L}(x^*x) = \lambda x^*x \quad (3)$$

$$\Leftrightarrow \langle (x^*x)^*v, w\rangle v - 2\langle u, x^*xu\rangle\langle v, w\rangle v + \langle v, w\rangle x^*xv = -\lambda x^*x \,\forall\, w \in c^n$$

for $w \perp v$, we have

$$\langle v, x^*xw\rangle v - 2\langle u, x^*xu\rangle.0 + 0.x^*xv = -\lambda x^*x$$

$$\langle v, x^*xw\rangle = -\lambda x^*xw \quad (4)$$

For $w = v$, we have

$$\langle v, x^*xv\rangle v - 2\langle u, x^*xu\rangle\langle v, v\rangle v + \langle v, v\rangle x^*xv = -\lambda x^*xw$$

since $\langle v, v\rangle = \|v\|^2 = 1$ we have

$$\langle v, x^*xv\rangle v - 2\langle u, x^*xu\rangle v + x^*xv = -\lambda x^*xv$$

$$(2\langle u, x^*xu\rangle - \langle v, x^*xv\rangle)v = x^*xv(1 + \lambda)$$

$$V_0 = Ker(L) = \{x \in M_n(c) : x(v^\perp) \subseteq v^\perp and x^*xv = \langle u, x^*xu\rangle v\}.$$

When λ = 0, (3) and (4) give

$$\mathbf{L}(x^*x) = 0 \quad (5)$$

$$\Leftrightarrow x^*xv = \langle v, x^*xv\rangle v \quad (6)$$

and

$$\langle v, x^*xw\rangle = 0 \text{ for } w \perp v \quad (7)$$

Then ker(L) is the space described before. In order to see that it coincides with V0, we can always use (6) and (7) for a purely imaginary λ and we easily conclude that L cannot have a non-null eigenvalues with null real part.

CONCLUSION

The result showed that the matrix space $M_n(c)=V_0 \oplus V_-$ will not decohere under N(T), i.e V_0 = N(T) implies $M_n(c)$ = N(T) since V− = {0}. Also by introducing GKSL of N(T) on $M_n(c), L_{(x^*x)} = 0$.

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