# Chapter 12

## On Hyers-Ulam Stability of Non-homogeneous Nonlinear Second Order Differential Equations

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## ABSTRACT

In this paper, Hyers-Ulam stability of nonhomogeneous nonlinear second order differential equations are considered. Conditions are stated to transform nonlinear second order differential equations to integral inequalities for easy application of Gronwall-Bellman-Bihari inequality. Our results improved and extended some known results in the literature.

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## INTRODUCTION

The purpose of this paper is to study Hyers-Ulam stability of the following equations:

$$u''(t) + a(t)u'(t) + b(t)u(t) + g(t)f(u(t)) = \kappa(t)\omega(u(t))u'(t) \tag{1.1}$$

and

$$u''(t) + \phi(t)\gamma(u(t))h(u'(t)) = \kappa(t)\omega(u(t))u'(t), \tag{1.2}$$

for all t > 0, with initial conditions

$$u(t_0) = u'(t_0) = 0, \tag{1.3}$$

where $a(t), b(t), \kappa(t), \varphi(t), g(t) \in C(\mathbf{R}_+)$, $f(u), \gamma(u), \omega(u), h(u) \in C(\mathbf{R}_+, \mathbf{R}_+)$.

The study of stability for various functional equations originated from a famous talk of Ulam, 1960 who posed a problem concerning the stability of functional equations:"Give conditions in order for a linear function near an approximately linear function to exist". Since then, this question has attracted the attention of many researchers. Note that the solution to this question was given by Hyers, 1941 for additive functions defined on Banach space in 1941. Thereafter, the result by Hyers 1941, was generalised by Rassias 1978, Aoki, 1950and Bourgin, 1988. After that, many authors extended the Ulam problem to other functional equations in various directions. In 1998, Alsina and Ger investigated the Hyers-Ulam stability of a differential equation. They have proved that for every differentiable mapping $f\colon \mathbf{I} \to \mathbf{R}$ satisfying $|f(x) - f(x)| \leq \in$ for every $x \in \mathbf{I}$l, where $\epsilon > 0$ is a given number and $\mathbf{I}$ is an open interval of $\mathbf{R}$, there exists a differentiable function $g\colon \mathbf{I} \to \mathbf{R}$ with property $g'(x) = g(x)$ and $|f(x) - g(x)| \leq 3 \in$ for all $x \in \mathbf{I}$. The result of Alsina and Ger, 1998 was extended by Miura *et al.*, 2003a; Miura *et al.*, 2003b , Takahasi *et al.*, 2002, and by Takahasi, *et al.*, 2004 to the Hyers-Ulam stability of the first order linear differential equations and linear differential equations of higher order with constant coefficients. Furthermore, Jung, 2006a; Jung, 2006b; Jung, 2004; Jung 2005 has obtained result on the stability of linear differential equations extending the results of Takahash *et al.,* 2004. Rus, 2010a; Rus, 2010b has proved some results on the stability of linear differential and integral equations using Gronwall's lemma and the technique of weakly Picard operators.

Recently, Wang, *et al.,* 2008 and Li & Shen 2010 proved the Hyers-Ulam stability of the linear differential equations of the first order and the linear differential equations of the second order with constant coefficients by using the method of integral factor.

Many of the authors prefer to consider the Hyers-Ulam stability of linear differential equations due to the fact that it can easily be handled. In this paper, equations (1.1) and (1.2) will be transformed to integral inequalities using the conditions that we shall prescribe for application of Gronwall-Bellman-Bihari inequality to study the Hyers- Ulam stability. The results established in this paper through Gronwall-Bellman-Bihari inequality extended some of the results in the literature.

# PRELIMINARY

First of all, we give some definitions, lemmas and theorems which are going to assistus in this work.

## Definition 1

We say that equation (1.1) has the Hyers-Ulam stability, if there existsa constant $K_1^* \geq 0$ with the following property: for every $\epsilon > 0$, $u(t) \in C^2(\mathbf{R}_+)$, if

$$|u''(t) + a(t)u'(t) + b(t)u(t) + g(t)f(u(t)) - \kappa(t)\omega(u(t))u'(t)| \leq \epsilon \qquad (2.1)$$

then, there exists some $u_0(t) \in C^2\mathbf{R}_+$ such that

$$|u(t) - u_0(t)| \leq K_1^*\epsilon.$$

We call such $K^*$ a Hyers-Ulam constant.

## Definition 2:

Equation (1.2) is Hyers-Ulam stable, if given $\epsilon > 0$ and there exists asolution $u(t) \in C^2(\mathbf{R}_+)$, such that

$$|u''(t) + \varphi(t)\gamma(u(t))h(u'(t)) - \kappa(t)\omega(u(t))u'(t)| \leq \epsilon, \qquad (2.2)$$

in addition, there exists positive $K_2^*$ and any solution $u_0(t) \in C^2\mathbf{R}_+$ of the (1.2) with initial condition (1.3) such that

$$|u(t) - u_0(t)| \leq K_2^*\epsilon,$$

We call such $K_2^*$ Hyers-Ulam constant for the differential equation (1.2).

## Definition 3:

A function $\omega: [0, \infty) \to [0, \infty)$ is said to belong to a class $\Psi$ if

i. $\omega(u)$ is nondecreasing and continuous for $u \geq 0$
ii. $\omega(u) \leq \omega\left(\frac{u}{v}\right)$for all $u$ and $v \geq 1$
iii. there exists a function $\varphi$, continuous on $[0, \infty)$ with $\omega(\alpha u) \leq \varphi(\alpha)\omega(u)$ for $\alpha \geq 0$

## Lemma 1: Bihari, 1957.

Let u(t), f(t) be positive continuous functions defined on $a \leq t \leq b, (\leq \infty)$ and $K > 0$, $M \geq 0$, further let ω(u) be a nonnegative nondecreasing continuous function for $u \geq 0$, then the inequality

$$u(t) \leq K + M \int_{t_0}^{t} f(s)\omega(u(s))ds, \;\; t_0 \leq t < b, \qquad (2.3)$$

implies the inequality

$$u(t) \leq \Omega^{-1}\left(\Omega(k) + M\int_{t_0}^{t} f(s)ds\right), \;\; t_0 \leq t \leq b' \leq b. \qquad (2.4)$$

Where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\omega(t)}, \qquad 0 < u_0 < u. \qquad (2.5)$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and $\Omega^{-1}$ is the inverse function of Ω and *t* must be in the subinterval $[t_0, b_0]$ of $[t_0, b]$ such that

$$\Omega(k) + M\int_{t_0}^{t} f(s)ds \in Dom(\Omega^{-1}).$$

## Theorem 1: (Murray, 1974)

If f(t) and g(t) are continuous in $[t0, t] \subseteq I$ and f(t) does not change sign in the interval, then there is a point $\xi \in [t0, t]$ such that $\int_{to}^{t} g(s)ds =$

$$g(\xi)\int_{t_0}^{t} f(s)ds$$

### Theorem 2: Fakunle and Arawomo, 2018a; Fakunle and Arawomo, 2018b; Fakunle and Arawomo, 2019, Fakunle and Arawomo, 2022a; Fakunle and Arawomo, 2022b

Suppose u(t), r(t), h(t) ∊ C(I, R+) and $(u), β(u) ∊ Ψ be nonnegative, monotonic, nondecreasing, continuous and ω(u) be a submultiplicative for u > 0.

Let

$$u(t) \leq K + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds \qquad (2.6)$$

for *K*, *T* and *L* positive constants, then

$$u(t) \leq \Omega^{-1}\left(\Omega(K) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{s} r(\alpha)d\alpha\right)\right)ds\right) F^{-1}\left(F(1) + T\int_{t_0}^{t} r(s)ds\right) \qquad (2.7)$$

where β(u) 6= $(u), Ω is defined in equation (2.5) and F(u) is defined as

$$F(u) = \int_{u_0}^{u} \frac{ds}{\beta(s)}, \quad 0 < u_0 \leq u, \qquad (2.8)$$

Where $F^{-1}$, $\Omega^{-1}$ are the inverses of F, Ω respectively and t is in the subinterval (0, b) ∈ **I** so that

$$F(1) + T\int_{t_0}^{t} r(s)ds \in Dom(F^{-1})$$

and

$$\Omega(K) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds \in Dom(\Omega^{-1})$$

## MAIN RESULTS

In this section, we establish the Hyers-Ulam stability of the nonlinear differential equations (1.1), (1.2) as follow: Theorem 3. Assume the following conditions

i $\displaystyle\int_{t_0}^{t} \frac{1}{b(s)}ds \leq p$, for $p > 0$ and all $t \in \mathbf{R}_+$, ii $\int_{t_0}^{t}\left(\frac{a(s)}{b(s)}-1\right)ds \le m, \text{ for } m > 0,$

iii $|u'(t)| \le \lambda, \text{ for } \lambda > 0,$

iv $\frac{g(t)}{b(t)} = \alpha(t), \ \frac{\kappa(t)}{b(t)} = \gamma(t),$

v $\lim_{t_0\to\infty}\int_{t_0}^{t}|u(s)|ds \le d < \infty, \ \text{where } d > 0,$

vi $\lim_{t_0\to\infty}\int_{t_0}^{t}\alpha(s)ds \le l < \infty, \ \text{where } l > 0,$

vii $\lim_{t_0\to\infty}\int_{t_0}^{t}\gamma(s)ds \le j < \infty, \ \text{where } j > 0,$

are satisfied and γ, α ∈ C(R+). In addition, let f, ω ∈ Ψ be continuous, nondecreasing and monotonic, then equation (1.1) has the Hyers-Ulam stability and Hyers-Ulam constant is given as

$$K_1^* = (m\lambda + d + p)\,\Omega^{-1}\left(\Omega(1) + \lambda j\omega\left(F^{-1}\left(F(1)+l\right)\right)\right)F^{-1}\left(F(1)+l\right) \qquad (3.1)$$

## Proof

Evaluating the inequality (2.1) we have

$$-\epsilon \le u''(t) + a(t)u'(t) + b(t)u(t) + g(t)f(u(t)) - \kappa(t)\omega(u(t))u'(t) \le \epsilon \qquad (3.2)$$

It is clear that

$$u''(t) + a(t)u'(t) + b(t)u(t) + g(t)f(u(t)) - \kappa(t)\omega(u(t))u'(t) \le \epsilon \qquad (3.3)$$

Let

$$E(t) = \frac{u'(t)}{b(t)} + u(t), \ b(t) \ne 0. \qquad (3.4)$$

We obtain

$$E(t) = E(t_0) + \int_{t_0}^{t}\frac{d}{ds}\left(\frac{u'(s)}{b(s)} + u(s)\right)ds, \qquad (3.5)$$

Using equation (1.3), we have

$$E(t_0)) = \frac{u'(t_0)}{b(t_0)} + u(t_0) = 0, \qquad (3.6)$$

It follows from equation (3.5) that

$$E(t) = \int_{t_0}^{t}\left(u'(s) + \frac{u''(s)}{b(s)} - \frac{d}{ds}b(s)\frac{u'(s)}{b^2(s)}\right)ds. \qquad (3.7)$$

Since *b(t)* is an increasing function, then d dsb(t) ≥ 0, equation (3.7) reduced to

$$E(t) = \int_{t_0}^{t} \left( u'(s) + \frac{u''(s)}{b(s)} \right) ds. \tag{3.8}$$

Substituting for u 00(t) in equation (3.8) by using equation (3.3), we have

$$\begin{aligned} E(t) \leq \int_{t_0}^{t} \Big( u'(s) - \frac{1}{b(s)} & \left( a(s)u'(s) + b(s)u(s) \right. \\ & \left. + g(s)f(u(s)) - \kappa(t)\omega(u(s))u'(s) - \epsilon \right) \Big) ds. \end{aligned} \tag{3.9}$$

Simplifying equation (3.9), we obtain

$$\begin{aligned} E(t) \leq \int_{t_0}^{t} \Big( (\frac{a(s)}{b(s)} - 1)u'(s) + u(s) + \frac{g(s)}{b(s)}f(u(s)) - \frac{\kappa(s)}{b(s)}\omega(u(s)) \\ - \frac{\epsilon}{b(s)} \Big) ds. \end{aligned} \tag{3.10}$$

It is clear from equation (3.4) that

$$u(t) \leq E(t) \tag{3.11}$$

Using inequality (3.11) in inequality (3.10), we have

$$\begin{aligned} u(t) \leq \int_{t_0}^{t} \Big( (\frac{a(s)}{b(s)} - 1)u'(s) + u(s) \\ + \frac{g(s)}{b(s)}f(u(s)) - \frac{\kappa(s)}{b(s)}\omega(u(s)) - \frac{\epsilon}{b(s)} \Big) ds. \end{aligned} \tag{3.12}$$

Expressing equation (3.12) further, we obtain

$$\begin{aligned} u(t) \leq \int_{t_0}^{t} (\frac{a(s)}{b(s)} - 1)u'(s)ds + \int_{t_0}^{t} u(s)ds \\ + \int_{t_0}^{t} \frac{g(s)}{b(s)}f(u(s))ds - \int_{t_0}^{t} \frac{\kappa(s)}{b(s)}\omega(u(s))ds - \epsilon \int_{t_0}^{t} \frac{1}{b(s)}ds. \end{aligned} \tag{3.13}$$

Taking the absolute value of both sides, we have

$$\begin{aligned} |u(t)| \leq \int_{t_0}^{t} (\frac{a(s)}{b(s)} - 1)|u'(s)|ds + \int_{t_0}^{t} |u(s)|ds \\ + \int_{t_0}^{t} \frac{g(s)}{b(s)}f(|u(s)|)ds + \int_{t_0}^{t} \frac{\kappa(s)}{b(s)}\omega(|u(s)|)ds + \epsilon \int_{t_0}^{t} \frac{1}{b(s)}ds. \end{aligned} \tag{3.14}$$

It follows from (3.14) that

$$
\begin{aligned}
|u(t)| \leq |u'(t)| \int_{t_0}^{t} (\frac{a(s)}{b(s)} - 1) ds + \int_{t_0}^{t} |u(s)| ds \\
+ \int_{t_0}^{t} \frac{g(s)}{b(s)} f(|u(s)|) ds + \int_{t_0}^{t} \frac{\kappa(s)}{b(s)} \omega(|u(s)|) ds + \epsilon \int_{t_0}^{t} \frac{1}{b(s)} ds.
\end{aligned}
\tag{3.15}
$$

Using the conditions (i-v), we get

$$
|u(t)| \leq m\lambda + d + p\epsilon + \int_{t_0}^{t} \alpha(s) f(|u(s)|) ds + \lambda \int_{t_0}^{t} \gamma(s) \omega(|u(s)|) ds. \tag{3.16}
$$

Simplifying equation (3.16) further we have

$$
|u(t)| \leq (m\lambda + d + p)\epsilon + \int_{t_0}^{t} \alpha(s) f(|u(s)|) ds + \lambda \int_{t_0}^{t} \gamma(s) \omega(|u(s)|) ds. \tag{3.17}
$$

Applying Theorem 2, we obtain

$$
\begin{aligned}
|u(t)| \leq (m\lambda + d + p)\, \epsilon \Omega^{-1} (\Omega(1) + \lambda \\
\int_{t_0}^{t} \gamma(s) \omega \left( F^{-1} \left( F(1) + \int_{t_0}^{s} \alpha(\delta) d\delta \right) \right) ds \right) F^{-1} \left( F(1) + \int_{t_0}^{t} \alpha(s) ds \right)
\end{aligned}
\tag{3.18}
$$

Applying the conditions (vi)- (vii), we arrive at

$$
\begin{aligned}
|u(t)| \leq (m\lambda + d + p)\, \epsilon \Omega^{-1} \left( \Omega(1) + \lambda j \omega \left( F^{-1} \left( F(1) + l \right) \right) \right) \\
F^{-1} \left( F(1) + l \right)
\end{aligned}
\tag{3.19}
$$

Hence,

$$
|u(t) - u(t_0)| \leq |u(t)| \leq K_1^* \epsilon
$$

Therefore,

$$
K_1^* = (m\lambda + d + p)\, \Omega^{-1} \left( \Omega(1) + \lambda j \omega \left( F^{-1} \left( F(1) + l \right) \right) \right) F^{-1} \left( F(1) + l \right).
$$

## Theorem 4:

Let u(t) ∈ C 2 (I, R+) satisfies the differential inequality (2.2) for all t ∈ I and for some  > 0, then there exists a solution u0(t) ∈ C 2 (I, R+) of equation (1.2) such that

$$
|u(t) - u_0(t)| \leq K_2^* \epsilon,
$$

provided the following conditions are satisfied:

i $G(u(t)) = \int_{u_0}^{u(t)} g(s) ds$

ii $\lim_{t_0 \to \infty} \int_{t_0}^{t} |u'(s)| ds \leq \Upsilon < \infty,$ where $\Upsilon > 0,$

iii $\phi(t)$ is nonincreasing then, $\frac{d}{ds}\phi(t) \geq 0$,

iv $\lim_{t_0\to\infty} \int_{t_0}^{t} \kappa(s)ds \leq \delta$, where $\delta > 0$

## Proof

From equation (2.2), we get

$$-\epsilon \leq u''(t) + \phi(t)\gamma(u(t))h(u'(t)) - \kappa(t)\omega(u(t))u'(t) \leq \epsilon. \tag{3.20}$$

It is clear that

$$u''(t) + \phi(t)\gamma(u(t))h(u'(t)) - \kappa(t)\omega(u(t))u'(t) \leq \epsilon. \tag{3.21}$$

Multiplying equation (3.21) by u 0 (t) and integrating from t₀ to t, we obtain

$$\begin{aligned} &\int_{t_0}^{t} u''(s)u'(s)ds + \int_{t_0}^{t} \phi(s)\gamma(u(s))h(u'(s))u'(s)ds \\ &\quad - \int_{t_0}^{t} \kappa(s)\omega(u(s))(u'(s))^2 ds \leq \epsilon \int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{3.22}$$

Using equation (1.3), we get

$$\begin{aligned} &\frac{1}{2}(u'(t))^2 + \int_{t_0}^{t} \phi(s)\gamma(u(s))h(u'(s))u'(s)ds \\ &- \int_{t_0}^{t} \kappa(s)\omega(u(s))(u'(s))^2 ds \leq \epsilon \int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{3.23}$$

By Theorem 1 there exists ι, µ ∈ [t0, t] such that

$$\begin{aligned} &\frac{1}{2}(u'(t))^2 + h(u'(\iota)) \int_{t_0}^{t} \phi(s)\gamma(u(s))u'(s)ds \\ &-(u'(\mu))^2 \int_{t_0}^{t} \kappa(s)\omega(u(s))ds \leq \epsilon \int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{3.24}$$

By condition (i) we obtain

$$\begin{aligned} &\frac{1}{2}(u'(t))^2 + h(u'(\iota)) \int_{t_0}^{t} \phi(s)\frac{d}{ds}G(u(s))ds \\ &-(u'(\mu))^2 \int_{t_0}^{t} \kappa(s)\omega(u(s))ds \leq \epsilon \int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{3.25}$$

Integrating by part using condition (iii) we have

$$\begin{aligned}\frac{1}{2}(u'(t))^2 + h(u'(\iota))\phi(t)G(u(t)) - (u'(\mu))^2 \int_{t_0}^{t} \kappa(s)\omega(u(s))ds \\ \leq \epsilon \int_{t_0}^{t} u'(s)ds.\end{aligned} \tag{3.26}$$

Put

$$\begin{aligned}h(u'(\iota))\phi(t)G(u(t)) \leq \epsilon \int_{t_0}^{t} u'(s)ds - \frac{1}{2}(u'(t))^2 \\ -(u'(\mu))^2 \int_{t_0}^{t} \kappa(s)\omega(u(s))ds.\end{aligned} \tag{3.27}$$

Taking the absolute value of both sides, we have

$$\begin{aligned}h(|u'(\iota)|)\phi(t)|G(u(t))| \leq \epsilon \int_{t_0}^{t} |u'(s)|ds + \frac{1}{2}(|u'(t)|)^2 \\ +(|u'(\mu)|)^2 \int_{t_0}^{t} \kappa(s)\omega(|u(s)|)ds.\end{aligned} \tag{3.28}$$

Using condition (iii) of Theorem 3 and condition (ii) of Theorem 4, we get

$$h(|u'(\iota)|)\phi(t)|G(u(t))| \leq \Upsilon\epsilon + \frac{1}{2}\lambda^2 + \lambda^2 \int_{t_0}^{t} \kappa(s)\omega(|u(s)|)ds. \tag{3.29}$$

Setting h(|u 0 (ι)|)φ(t)|G(u(t))| ≥ |u(t)|, we obtain

$$|(u(t))| \leq \epsilon \left(\Upsilon + \frac{1}{2}\lambda^2\right) + \lambda^2 \int_{t_0}^{t} \kappa(s)\omega(|u(s)|)ds. \tag{3.30}$$

Applying Theorem 1

$$|u(t|) \leq \epsilon \left(\Upsilon + \frac{1}{2}\lambda^2\right) \Omega^{-1} \left(\Omega(1) + \lambda^2 \int_{t_0}^{t} \kappa(s)ds\right). \tag{3.31}$$

Using the condition (iv), we obtain

$$|u(t|) \leq \epsilon \left(\Upsilon + \frac{1}{2}\lambda^2\right) \Omega^{-1} \left(\Omega(1) + \lambda^2\delta\right). \tag{3.32}$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_2^*\epsilon$$

Therefore,

$$K_2^* = \left(\Upsilon + \frac{1}{2}\lambda^2\right) \Omega^{-1} \left(\Omega(1) + \lambda^2\delta\right)$$

$\square$

### Example 1:

Consider the following equation

$$u''(t) + t^2 u'(t) + t^2 u(t) + t^{-6} u^4(t) = t^{-3} u^2(t) u'(t), \quad t > 0,$$

Where $b(t) = t^4,\ a(t) = t^2,\ \frac{g(t)}{b(t)} = \frac{t^{-6}}{t^4} = t^{-10},\ \frac{\kappa}{b(t)} = \frac{t^{-3}}{t^4} = t^{-7}$ the criteria of Theorem 3, we arrive at the result.

### Example 2:

Consider the following equation

$$u''(t) + t^{-3} u^2(t) u'(t) = t^{-3} u^2(t) u'(t), \quad t > 0,$$

Using the criteria of Theorem 4, we arrive at the result.

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