Chapter 7

Symmetry Factorization of Lamé Equation

Idiong U.S.
Corresponding author's email: usidiong@gmail.com

ABSTRACT

The study of Lamé operator remains an open problem because of its rich symmetry and other algebraic properties. One of the essential tools that is used in its study is the factorization technique. This paper gives some of the answers to the questions that arise in the consideration of the integration of the Lamé operator equation on the elliptic curve using infinitesimal transformation.

Keywords: Elliptic, Factorisation, Infinitesimal transformation, Lie symmetry, Lame equation

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INTRODUCTION

Lam´e equation in its different forms are second order ordinary differential equations in the complex domain. They appear in literatures as Fuchsian differential equations with four regular singularities $e_1 = \wp(\omega_1), e_2 = \wp(\omega_2), e_3 = \wp(\omega_3), e_4 = \infty$ where $\omega_2 = \frac{\omega_1 + \omega_3}{2}$ (see Churchill 1989). In its compact Weierstrass form

$$\left\{ \left(\frac{\mathrm{d}}{\mathrm{d}u} \right)^2 - \ell(\ell+1)\wp(u; g_2, g_3) - B \right\} \Psi(u) = 0, \quad \ell \in \mathbb{C} \qquad (1.1)$$

is defined on the family of elliptic curve

$$E_{g_1,\, g_2} := \left\{ y | y^2 = 4w^3 - g_2 w - g_3 = 4 \prod_{i=1}^{3} (w - e_i)^3 \right\}$$

where B is the accessory parameter which plays the role of the eigenvalue of equation (1.1) and $\wp$(u) is the Weierstrass elliptic $\wp$-function. In the selfadjoint Fuchsian form

$$4\sqrt{\varphi(t)}\frac{d}{dt}\left(\sqrt{\varphi(t)}\frac{d}{dt}\Psi(t)\right) = (\ell(\ell+1)t+B)\Psi(t), \tag{1.2}$$

where, $\varphi(t) = (a^2+t)(b^2+t)(c^2+t) = (\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3) = (a^2-b^2)^2\,\mathrm{sn}^2\alpha\,\mathrm{cn}^2\alpha\,\mathrm{dn}^2\alpha$, and $(a^2-b^2)^2 \in \mathbb{R}$ (see Wang et. al 1986, pp. 576-580, §11.1). Here, t can assume any variable λ, μ, v in the coordinates of the ellipsoid.

The major existing technique for solving the Lam´e equation is the operator factorization which gives elliptic solutions. The Ricatti equation plays an intermediary role in the study of Supersymmetry (SUSY) factorization as well as Lie symmetry analysis of Schr¨odinger operators. Hence, the need to understand a technique of solving Riccati equation and its Lie symmetry considerations cannot be overemphasized. The paper shall be outlined as follows: section 2 deals with infinitesimal transformation and the solution of Ricatti equations; section 3 deals with the Lie symmetry of Ricatti equations and section 4 deals with our main results.

INFINITESIMAL TRANSFORMATION AND THE SOLUTION OF RICATTI EQUATIONS

In this section, we examine the group theoretical approach of solving ordinary differential equations using infinitesimal transformations.

2.1 Theorem (Hill 1992, p. 31). Consider the generalized Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}w} + p(w)y = q(w) + r(w)y^2$$

which remains invariant under the transformation of the form

$$w_1 = f(w,\epsilon), \qquad\qquad y_1 = g(w,\epsilon)y$$

provided $r(w) = q(w)s(w)^2$. Then the infinitesimal displacement functions

$$\xi(w)=\frac{1}{q(w)s(w)},\eta(w)=-\frac{p(w)}{q(w)s(w)}$$

with suitable canonical coordinates

$$u(w,y)=s(w)y,\quad v(w,y)=\int_{\Xi_{w_0,w}}s(t)q(t)\mathrm{d}t$$

reduces the Ricatti differential equation to a solvable form

$$\frac{\mathrm{d}u}{\mathrm{d}v}=1+u^2$$

so that the solution of the Ricatti differential equation is

$$y(w)=\frac{1}{s(w)}\tan\left(\int_{\Xi_{w_0,w}}q(t)s(t)\mathrm{d}t+C\right).$$

Proof. Now, given the generalized Ricatti equation

$$\frac{\mathrm{d}y}{\mathrm{d}w}+p(w)y=q(w)+r(w)y^2,$$

let

$$w_1=f(w,\epsilon),\qquad y_1=g(w,\epsilon)y.\tag{2.1}$$

By one-parameter group infinitesimal transformation

$$w_1=w+\epsilon\xi(w)+O(\epsilon^2),\qquad y_1=y+\epsilon\eta(w)y+O(\epsilon^2).\tag{2.2}$$

Taking the derivative of equation (2.2) we have

$$\mathrm{d}w_1=\mathrm{d}w+\epsilon\xi'(w)\mathrm{d}w+O(\epsilon^2),\qquad \mathrm{d}y_1=\mathrm{d}y+\epsilon[\eta'(w)y(w)\mathrm{d}w+\eta(w)\mathrm{d}y]+O(\epsilon^2).\tag{2.3}$$

Now taking the quotients of parametric derivatives in (2.3) we obtain by first prolongation formula (see Gilmore, 2008,§16.2.2, p.287)

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}w_1}&=\frac{\mathrm{d}y+\epsilon[\eta'(w)y(w)\mathrm{d}w+\eta(w)\mathrm{d}y]}{\mathrm{d}w+\epsilon\xi'(w)\mathrm{d}w}+O(\epsilon^2)\\&=\frac{\frac{\mathrm{d}y}{\mathrm{d}w}+\epsilon[\eta'(w)y(w)+\eta(w)\frac{\mathrm{d}y}{\mathrm{d}w}]}{1+\epsilon\xi'(w)}+O(\epsilon^2)\\&=\left\{\frac{\mathrm{d}y}{\mathrm{d}w}+\epsilon\big[\eta'(w)y(w)+\eta(w)\frac{\mathrm{d}y}{\mathrm{d}w}]\right\}\left\{1+\epsilon\xi'(w)\right\}^{-1}+O(\epsilon^2)\\&=\frac{\mathrm{d}y}{\mathrm{d}w}+\epsilon\big[\eta'(w)y(w)+\eta(w)\big]\frac{\mathrm{d}y}{\mathrm{d}w}+O(\epsilon^2).\end{aligned}\tag{2.4}$$

as $\epsilon^2\to 0$. Next, we show that the generalized Ricatti equation

$$\frac{\mathrm{d}y}{\mathrm{d}w} + p(w)y = q(w) + r(w)y^2, \qquad (2.5)$$

remains invariant under the transformation (2.5) provided r(w) = q(w)s(w) ^{2} i.e.

$$\frac{\mathrm{d}y_1}{\mathrm{d}w_1} + p(w_1)y_1 = q(w_1) + r(w_1)y_1^2. \qquad (2.6)$$

Now, from (2.5) we have

$$\mathrm{d}w_1 = f'(w)\mathrm{d}w, \qquad \mathrm{d}y_1 = yg'(w)\mathrm{d}w + g(w)dy \qquad (2.7)$$

so that taking the differential quotient we have

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}w_1} &= \frac{yg'(w)\mathrm{d}w + gdy}{f'(w)\mathrm{d}w} \\ &= \frac{g'(w)}{f'(w)}y + \frac{g(w)}{f'(w)}\frac{\mathrm{d}y}{\mathrm{d}w}. \qquad (2.8) \end{aligned}$$

Now, substituting (2.5) and (2.8) into (2.6), we have

$$\frac{g'(w)}{f'(w)}y + \frac{g(w)}{f'(w)}\frac{\mathrm{d}y}{\mathrm{d}w} + p(f(w))g(w)y = q(f(w)) + r(f(w))g(w)^2y^2. \qquad (2.9)$$

Multiplying through (2.9) by $\frac{f\prime(w)}{g(w)}$, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}w} + \left[\frac{g'(w)}{g(w)} + p(f(w))f'(w)\right]y = q(f(w))\frac{f'(w)}{g(w)} + r(f(w))f'(w)g(w)y^2. \qquad (2.10)$$

Now, comparing (2.5) with (2.10) we have

$$\begin{aligned} p(w) &= \frac{g'(w)}{g(w)} + p(f(w))f'(w), \qquad (2.11) \\ q(w) &= q(f(w))\frac{f'(w)}{g(w)}, \qquad (2.12) \\ r(w) &= r(f(w))f'(w)g(w)y^2. \qquad (2.13) \end{aligned}$$

Now setting,

$$\begin{aligned} f(w) &= w + \epsilon\xi(w) + O(\epsilon^2), \qquad (2.14)\\ g(w) &= 1 + \epsilon\eta(w) + O(\epsilon^2), \qquad (2.15)\\ p(f(w)) &= p(w + \epsilon\xi(w) + O(\epsilon^2))\\ &= p(w) + \epsilon\xi(w)p'(w) + O(\epsilon^2), \qquad (2.16) \end{aligned}$$

and similarly,

$$q(f(w)) = q(w) + \epsilon\xi(w)q'(w) + O(\epsilon^2). \tag{2.17}$$

Thus from (2.11)-(2.17) we have

$$\begin{aligned} p(w) &= \frac{g'(w)}{g(w)} + p(f(w))f'(w)\\ &= \frac{\epsilon\eta'(w)}{1+\epsilon\eta(w)} + (p(w) + \epsilon\xi(w)p'(w))(1+\epsilon\xi'(w)) + O(\epsilon^2)\\ &= \epsilon\eta'(w)(1+\epsilon\eta(w))^{-1} + p(w) + \epsilon\xi'(w)p(w) + \epsilon\xi(w)p'(w) + O(\epsilon^2)\\ &= \epsilon\eta'(w)(1-\epsilon\eta(w)) + p(w) + \epsilon\xi'(w)p(w) + \epsilon\xi(w)p'(w) + O(\epsilon^2)\\ &= \epsilon\eta'(w) + p(w) + \epsilon\xi'(w)p(w) + \epsilon\xi(w)p'(w) + O(\epsilon^2)\\ &= p(w) + \epsilon[\eta'(w) + \xi'(w)p(w) + \xi(w)p'(w)] + O(\epsilon^2)\\ \Rightarrow \eta'(w) &+ \xi'(w)p(w) + \xi(w)p'(w) = 0 \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} q(w) &= q(f(w))\frac{f'(w)}{g(w)}\\ &= (q(w) + \epsilon\xi(w)q'(w))\left(\frac{1+\epsilon\xi'(w)}{1+\epsilon\eta(w)}\right) + O(\epsilon^2)\\ &= (q(w) + \epsilon\xi(w)q'(w))\,(1+\epsilon\xi'(w))\,(1-\epsilon\eta(w)) + O(\epsilon^2)\\ &= (q(w) + \epsilon\xi(w)q'(w))(1 - \epsilon\eta(w) + \epsilon\xi'(w)) + O(\epsilon^2)\\ &= q(w) + \epsilon[(\xi'(w) - \eta(w))q(w) + \xi(w)q'(w)] + O(\epsilon^2)\\ \Rightarrow (\xi'(w) &- \eta(w))q(w) + \xi(w)q'(w) = 0. \end{aligned} \tag{2.19}$$

Also,

$$\begin{aligned}
r(w) &= r(f(w))f'(w)g(w)y^2 \\
&= (r(w)+\epsilon\xi(w)r'(w))(1+\epsilon\xi'(w))(1+\epsilon\eta(w))+O(\epsilon^2) \\
&= (r(w)+\epsilon\xi(w)r'(w))(1+\epsilon(\xi'(w)+\eta(w)))+O(\epsilon^2) \\
&= r(w)+\epsilon[\xi(w)r'(w)+\xi'(w)r(w)+\eta(w)r(w)]+O(\epsilon^2) \\
\Rightarrow\ & \xi(w)r'(w)+\xi'(w)r(w)+\eta(w)r(w)=0.
\end{aligned} \tag{2.20}$$

Now equations (2.11)-(2.13) can now be rewritten respectively as

$$\eta'(w)+(\xi(w)p(w))' = 0, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}w} \tag{2.21}$$

$$\frac{\eta(w)}{\xi(w)} = \frac{\xi'(w)}{\xi(w)}+\frac{q'(w)}{q(w)} \tag{2.22}$$

$$\frac{r'(w)}{r(w)}+\frac{\xi'(w)}{\xi(w)}+\frac{\eta(w)}{\xi(w)} = 0. \tag{2.23}$$

Substituting (2.22) in (2.23) we have

$$\frac{r'(w)}{r(w)}+2\frac{\xi'(w)}{\xi(w)}+\frac{q'(w)}{q(w)}=0. \tag{2.24}$$

Integrating (2.24) we have

$$\begin{aligned}
\log_e\left[r(w)\xi^2(w)q(w)\right] &= \log_e C_1 \\
\Rightarrow r(w)\xi^2(w)q(w) &= C_1.
\end{aligned} \tag{2.25}$$

Also integrating (2.21) we obtain

$$\eta(w)+\xi(w)p(w)=C_2.$$

Now, let $C_2 = 0$, then we have that

$$\frac{\eta(w)}{\xi(w)}=-p(w). \tag{2.26}$$

Substituting (2.26) into (2.22) it is easily seen that we have

$$\frac{\xi'(w)}{\xi(w)}=-p(w)-\frac{q'(w)}{q(w)}, \tag{2.27}$$

which on integrating gives us

$$\begin{aligned}\log_e \xi(w) &= -\int_{\Xi_{w_0,w}} p(t)\mathrm{d}t - \log_e q(w)\\ \Rightarrow \xi(w) &= \frac{1}{q(w)}\exp\left(-\int_{\Xi_{w_0,w}} p(t)\mathrm{d}t\right)\\ &= \frac{1}{q(w)s(w)}, \qquad (2.28)\end{aligned}$$

where

$$s(w) = \exp\left(\int_{\Xi_{w_0,w}} p(t)\mathrm{d}t\right).$$

Now from (2.26) and (2.28) we have

$$\begin{aligned}\eta(w) &= -p(w)\xi(w)\\ &= -p(w)\frac{1}{q(w)s(w)}\\ &= -\frac{p(w)}{q(w)s(w)}. \qquad (2.29)\end{aligned}$$

Furthermore, substituting (2.27) into (2.24) we have

$$\begin{aligned}\frac{r'(w)}{r(w)} + 2\left(-p(w) - \frac{q'(w)}{q(w)}\right) + \frac{q'(w)}{q(w)} &= 0\\ \Rightarrow \frac{r'(w)}{r(w)} - 2p(w) - \frac{q'(w)}{q(w)} &= 0. \qquad (2.30)\end{aligned}$$

On integrating (2.30) we arrive

$$\begin{aligned}r(w) &= q(w)\exp\left(2\int_{\Xi_{w_0,w}} p(t)\mathrm{d}t\right)\\ &= q(w)s^2(w) \qquad (2.31)\end{aligned}$$

If we now choose our canonical coordinates to be

$$\begin{aligned}u(w,y) &= s(w)y, \qquad v(w,y) = \int_{\Xi_{w_0,w}} s(t)q(t)\mathrm{d}t, \qquad (2.32)\\ \mathrm{d}u &= (s'(w)y + y's(w))\mathrm{d}w, \qquad (2.33)\\ \mathrm{d}v &= s(w)q(w)\mathrm{d}w, \qquad (2.34)\end{aligned}$$

so that taking the quotient of (2.33) by (2.34)

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}v} &= \frac{s'(w)y(w)+y'(w)s(w)}{s(w)q(w)}\\
&= \frac{s'(w)}{s(w)}\frac{y}{q(w)}+\frac{y'}{q(w)}\\
&= \frac{s'(w)}{s(w)}\frac{y}{q(w)}+\frac{1}{q(w)}\left[-p(w)y+q(w)+r(w)y^2\right]\\
&= \frac{s'(w)}{s(w)}\frac{y}{q(w)}+1-\frac{p(w)y}{q(w)}+\frac{r(w)}{q(w)}y^2\\
&= \frac{s'(w)}{s(w)}\frac{y}{q(w)}+\eta(w)s(w)y+1+s^2(w)y^2 \qquad \text{(from (2.29) and (2.31))}\\
&= \left[\frac{s'(w)}{s(w)q(w)}+\eta(w)s(w)\right]y+1+s^2(w)y^2\\
&= [\xi(w)s'(w)+\eta(w)s(w)]y+1+s^2(w)y^2 \qquad \text{(from (2.28))}\\
&= 0+1+s^2(w)y^2 \qquad \left(\because \frac{\mathrm{d}(\log_e s(w))}{\mathrm{d}w}=\frac{s'(w)}{s(w)}=p(w)=-\frac{\eta(w)}{\xi(w)}\right).\\
&= 1+u^2.
\end{aligned}
$$

By variable separable method we have

$$\frac{\mathrm{d}u}{1+u^2}=\mathrm{d}v.$$

Integrating both sides, we have

$$
\begin{aligned}
\tan^{-1}u &= v+c\\
\Rightarrow u &= \tan(v+c)\\
\Rightarrow s(w)y &= \tan\left(\int_{\Xi_{w_0,w}} s(t)q(t)\mathrm{d}t+c\right)\\
\Rightarrow y &= \frac{1}{s(w)}\tan\left(\int_{\Xi_{w_0,w}} s(t)q(t)\mathrm{d}t+c\right). \qquad (2.35)
\end{aligned}
$$

The result is obtained as required.

The above result holds where $w, w_0 \in \mathbb{C}\backslash[0,\infty)$are two endpoints which are connected by simple rectifiable curve $\Xi_{w_0,w}$.

LIE SYMMETRIES

In this section, we discuss the Lie symmetries of differential operators which is an important tool for the study of the group properties of a linear differential operator. We know (Cariñena & Ramos, 1998, p.3) that Ricatti equation of the form

$$\frac{\mathrm{d}z(w)}{\mathrm{d}w} = a_0(w) + a_1(w)z(w) + a_2(w)z^2(w), \qquad z(w) = x(w) + iy(w) \tag{3.1}$$

can be considered as a differential equation determining the integral curves of w-dependent vector fields

$$X_w = \left(a_0(w) + a_1(w)z + a_2(w)z^2\right)\frac{\partial}{\partial z}, \qquad z = z(w) \tag{3.2}$$

so that X_{ω} is the linear combination with ω-dependent coefficients of the three vector fields

$$\mathcal{J} = \frac{\partial}{\partial z}, \qquad \mathcal{J}_+ = z\frac{\partial}{\partial z}, \qquad \mathcal{J}_- = z^2\frac{\partial}{\partial z}. \tag{3.3}$$

Now the Riccati equation associated to Brioschi-Halphen equation (BHE)

$$\left\{(4w^3 - g_2w - g_3)D^2 + (1-2n)\left(6w^2 - \frac{1}{2}g_2\right)D + 4n(2n-1)w - 4B\right\}\Psi = 0, \quad D = \frac{\mathrm{d}}{\mathrm{d}w}. \tag{3.4}$$

can be obtained by setting

$$z(w) = \frac{\Psi'(w)}{\Psi(w)}.$$

Thus, the associated Riccati equation is given by

$$\frac{\mathrm{d}z}{\mathrm{d}w} + \frac{4n(n-1)w - 4B}{4w^3 - g_2w - g_3} + \frac{(1-2n)(6w^2 - g_2)}{4w^3 - g_2w - g_3}z + z^2 = 0.$$

Here, $z = z(w)$.. This implies that the differential equation determining the integral curves of w-dependent vector fields can be written as

$$X_w := \left(-\frac{4n(n-1)w - 4B}{4w^3 - g_2w - g_3} - \frac{(1-2n)(6w^2 - g_2)}{4w^3 - g_2w - g_3}z - z^2\right)\frac{\partial}{\partial z}.$$

Now let $a_1(w), a_2(w), a_3(w)$ be functions related to the family elliptic curves E_{g_2,g_3} which are expressed by

$$\begin{aligned}
a_0(w) &= -\frac{4n(n-1)w - 4B}{4w^3 - g_2w - g_3} \equiv -\sum_{j=0}^{\infty}\left(\nu(1,j) + \nu(2,j) + \nu(3,j)\right)w^j; \\
a_1(w) &= -\frac{(1-2n)(6w^2 - g_2)}{4w^3 - g_2w - g_3} = -\sum_{j=0}^{\infty}\left(\mu(1,j) + \mu(2,j) + \mu(3,j)\right)w^j; \text{and} \\
a_2(w) &= 1.
\end{aligned}$$

Here,

$$\nu(s,j) = -\frac{n(n-1)e_s - B}{(e_s - e_t)(e_s - e_{t'})} e_s^{-j-1},$$

and

$$\mu(s,j) = -\frac{(1-2n)(6e_s^2 - g_2)}{4(e_s - e_t)(e_s - e_{t'})} e_s^{-j-1}$$

where $s, t, t' = 1,2,3, s \neq t$ and $s \neq t'$

The basis of the vector fields Xω made up of $\{\mathcal{J}\ \mathcal{J}_+\ \mathcal{J}_-\}$ are the generators of a Vessiot-Guldberg Lie algebra, $\langle \mathcal{J}\ \mathcal{J}_+\ \mathcal{J}_- \rangle \simeq \mathfrak{sl}(2, \mathbb{C}$, which is made up of traceless matrices having basis

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (3.5)$$

with Lie algebraic commutation

$$[H, E_+] = E_+, \quad [E_-, H] = E_-, \quad [E_+, E_-] = 2H. \qquad (3.6)$$

We remark here that the matrices H, E+, E− in (3.6) commute exactly as the $\mathcal{J}\ \mathcal{J}_+\ \mathcal{J}_-$ in (3.3). It is also observed that $\mathcal{J}$ and $\mathcal{J}_+$ generate a 2-dimensional Lie subalgebra isomorphic to the Lie algebra of the affine group of transformation in one dimension and same holds for $\mathcal{J}_+$ and $\mathcal{J}_-$. The one parameter subgroups of local transformation of C generated by $\mathcal{J}$, $\mathcal{J}_+$ and $\mathcal{J}_-$ are respectively

- Translation $\omega \mapsto \omega + \epsilon$
- Dilation $\omega \mapsto e^{\epsilon}\ \omega$
- Infinitesimal/Mobios $\omega \mapsto \frac{\omega}{1-\omega\epsilon}$

for which $\mathcal{J}_-$ is not a complete vector field of C. However, a one point compactification of $\mathbb{C}$ is practicable and then $\mathcal{J}$, $\mathcal{J}_+$ and $\mathcal{J}$, could be considered as the fundamental vector fields corresponding to the action of $SL\,(2, \mathbb{C})$ on the complete complex rectifiable curve. $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ given by

$$\begin{aligned} \Omega(A,w) &= \frac{\alpha w+\beta}{\gamma w+\delta}, \quad \text{if} \quad x \neq -\frac{\delta}{\gamma} \\ \Omega(A,\infty) &= \frac{\alpha}{\gamma}, \quad \Omega(A,-\frac{\delta}{\gamma}) = \infty, \end{aligned} \tag{3.7}$$

where,

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C}).$$

Alternatively, the vector field Xw in (3.2) can also be written as

$$X_w = a_0(w)X_1 + a_1(w)X_2 + a_2(w)X_3,$$

where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad (x^2-y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}$$

which also span the Vessiot-Guldberg Lie algebra of vector fields $V \simeq s\ell(2,\mathbb{C})$ which obeys the commutation relation

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3.$$

Hence, $\{X\omega\}_{\omega\in\mathbb{CP}^1} \subseteq VX \subseteq V$ and V^X is finite-dimensional, which makes X into a Lie system. Hence, it admits a superposition rule. This is an analogy to what is obtained in (Munoz, 2015).

MAIN RESULT

Next, we compute the eigenfunctions of the Lam´e equation using the infinitesimal transformation of its associated Riccati equation. Now having obtained a quadrature (2.35) we now apply it to factorization of the Lamé operator equation. The Lamé equation obtained from ellipsoidal harmonics, takes the form

$$\varphi(\lambda)\frac{\mathrm{d}^2\psi_r(\lambda)}{\mathrm{d}\lambda^2} + \frac{1}{2}\frac{\mathrm{d}\psi_r(\lambda)}{\mathrm{d}\lambda} = (K\lambda + C)\psi_r(\lambda), \quad r = 1,2,3. \tag{4.1}$$

Here $\varphi(t) = (a^2+\lambda)(b^2+\lambda)(c^2+\lambda) = (\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3) = (a^2-b^2)^2 sn^2\alpha cn^2\alpha dn^2\alpha$, $K = n(n+1)$ and $(a^2-b^2)^2 \in \mathbb{R}$. If without loss of generality we set ψr ≡ ψ and the Lam´e operator equation (4.1) in self adjoint form takes the form

$$\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)\psi(t)+\frac{(n(n+1)t+E)}{4}\psi(t)=0, \qquad (4.2)$$

Proposition 4.1

The Lamé equation (4.1) factors as

$$\left[\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{1}{k_1}\tan\left(k_1\int_{\Xi_{t_0,t}}\frac{(n(n+1)\mathrm{d}\xi+E)}{4\sqrt{\varphi(\xi)}}\mathrm{d}\xi+k_2\right)\right]$$
$$\times\left[\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}-\frac{1}{k_1}\tan\left(k_1\int_{\Xi_{t_0,t}}\frac{(n(n+1)\xi+E)}{4\sqrt{\varphi(\xi)}}\mathrm{d}\xi+k_2\right)\right]\psi(t)=0,$$

having the eigenfunctions

$$\psi_n^{(+)}(t,E)=\exp\left\{-\int_{\Xi_{\tau,t}}\frac{1}{k_1\sqrt{\varphi(\tau)}}\tan\left(k_1\int_{\Xi_{\tau_0,\tau}}\frac{(n(n+1)\xi+E)}{4\sqrt{\varphi(\xi)}}\mathrm{d}\xi+k_2\right)\mathrm{d}\tau\right\}.$$

Proof. Let the equation (4.2) factor as

$$\left[\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}+y(t)\right]\left[\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}-y(t)\right]\psi(t)=0, \qquad (4.3)$$

which on multiplying out gives us

$$\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)\psi(t)-\sqrt{\varphi(t)}y'(t)\psi(t)$$
$$-\sqrt{\varphi(t)}y(t)\frac{\mathrm{d}\psi}{\mathrm{d}t}+y(t)\sqrt{\varphi(t)}\frac{\mathrm{d}\psi}{\mathrm{d}t}-y^2(t)\psi=0$$
$$\Rightarrow\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)\psi(t)-\left(\sqrt{\varphi(t)}y'(t)+y^2(t)\right)\psi=0 \qquad (4.4)$$

Now comparing (4.2) with (4.4) we have

$$\sqrt{\varphi(t)}y'(t)+y^2(t)-\frac{(n(n+1)t+E)}{4}=0$$

which can be rewritten as

$$y'(t)=\frac{(n(n+1)t+E)}{4\sqrt{\varphi(t)}}-\frac{1}{\sqrt{\varphi(t)}}y^2(t) \qquad (4.5)$$

Comparing (4.5) with the generalized Riccati equation in (2.5) we obtain

$$p(t)=0,\qquad q(t)=\frac{(n(n+1)t+E)}{4\sqrt{\varphi(t)}},\qquad r(t)=-\frac{1}{\sqrt{\varphi(t)}}$$

so that,

$$s(t)=\exp\left(\int_{\Xi_{t_0,t}}p(\zeta)\mathrm{d}\zeta\right)=\text{constant}\qquad k_1(\text{say})\tag{4.6}$$

From (2.35) we have

$$y_n(t)=\frac{1}{k_1}\tan\left(k_1\int_{\Xi_{t_0,t}}\frac{(n(n+1)\zeta+E)}{4\sqrt{\varphi(\zeta)}}\mathrm{d}\zeta+k_2\right),\qquad n\in\mathbb{N}\tag{4.7}$$

(4.7) holds for the case t = λ, ν. For the case t = μ we replace φ(t) with −φ(t), we have

$$y_n(\mu)=\frac{1}{k_1}\tan\left(k_1\int_{\Xi_{\mu_0,\mu}}\frac{(n(n+1)\zeta+E)}{4\sqrt{-\varphi(\zeta)}}\mathrm{d}\zeta+k_2\right),\qquad n\in\mathbb{N}\tag{4.8}$$

where (4.7) and (4.8) gives us the super potential for the operator (4.2) So that (4.3) now becomes

$$\begin{aligned}&\left[\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{1}{k_1}\tan\left(k_1\int_{\Xi_{t_0,t}}\frac{(n(n+1)\zeta+E)}{4\sqrt{\varphi(\zeta)}}\mathrm{d}\zeta+k_2\right)\right]\\&\qquad\times\left[\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}-\frac{1}{k_1}\tan\left(k_1\int_{\Xi_{t_0,t}}\frac{(n(n+1)\zeta+E)}{4\sqrt{\varphi(\zeta)}}\mathrm{d}\zeta+k_2\right)\right]\psi(t)=0\end{aligned}\tag{4.9}$$

Now, the eigenfunctions $\psi\equiv\psi_n^{(-)}(t,E)$ are obtained by solving the 1st order linear differential equation

$$\left[\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}-\frac{1}{k_1}\tan\left(k_1\int_{\Xi_{t_0,t}}\frac{(n(n+1)\zeta+E)}{4\sqrt{\varphi(\zeta)}}\mathrm{d}\zeta+k_2\right)\right]\psi_n^{(-)}(t,E)=0,$$

which can be rewritten in the form

$$\frac{1}{\psi_n^{(-)}(t,E)}\frac{\mathrm{d}\psi_n^{(-)}(t,E)}{dt}=\frac{1}{k_1\sqrt{\varphi(t)}}\tan\left(k_1\int_{\Xi_{t_0,t}}\frac{(n(n+1)\zeta+E)}{4\sqrt{\varphi(\zeta)}}\mathrm{d}\zeta+k_2\right)\tag{4.10}$$

Now, we integrate both sides of (4.10) to obtain

$$\begin{aligned}\log_e \psi_n^{(-)}(t,E) &= \int_{\Xi_{\tau,t}} \frac{1}{k_1\sqrt{\varphi(\tau)}} \tan\left(k_1 \int_{\Xi_{\tau_0,\tau}} \frac{(n(n+1)\zeta+E)}{4\sqrt{\varphi(\zeta)}} \mathrm{d}\zeta + k_2\right) \mathrm{d}\tau \\ \Rightarrow \psi_n^{(-)}(t,E) &= \exp\left\{\int_{\Xi_{\tau,t}} \frac{1}{k_1\sqrt{\varphi(\tau)}} \tan\left(k_1 \int_{\Xi_{\tau_0,\tau}} \frac{(n(n+1)\zeta+E)}{4\sqrt{\varphi(\zeta)}} \mathrm{d}\zeta + k_2\right) \mathrm{d}\tau\right\}\end{aligned} \tag{4.11}$$

where, $\tau \in [0,t]$. Similarly, the eigenfunctions $\psi_n^{(+)}$which satisfy the 1st order linear operator equation

$$\left[\sqrt{\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{k_1}\tan\left(k_1 \int_{\Xi_{t_0,t}} \frac{(n(n+1)\zeta+E)}{4\sqrt{\varphi(\zeta)}} \mathrm{d}\zeta + k_2\right)\right]\psi_n^{(+)}(t,E) = 0, \tag{4.12}$$

are obtained as

$$\psi_n^{(+)}(t,E) = \exp\left\{-\int_{\Xi_{\tau,t}} \frac{1}{k_1\sqrt{\varphi(\tau)}} \tan\left(k_1 \int_{\Xi_{t_0,\tau}} \frac{(n(n+1)\zeta+E)}{4\sqrt{\varphi(\zeta)}} \mathrm{d}\zeta + k_2\right) \mathrm{d}\tau\right\} \tag{4.13}$$

In the cases (4.11) and (4.13), t = λ, ν, analogously we obtain the eigenfunctions in terms of t = μ by replacing φ(t) by −φ(t) for which we obtain hyperbolic trigonometric solutions. In each case we obtain a sequence of eigenfunctions $\left\{\psi_n^{(-)}(t,E)\right\}$, $\left\{\psi_n^{(+)}(t,E)\right\} \in \mathfrak{H}_t,\ t=\lambda,\mu,\nu$ with n ∈ N. The sequence generated allow for consideration of completeness of the Hilbert space.

CONCLUSION

The result obtained in this paper shows a new alternative technique of solving second order differential equations and in particular the Lamé equation through a proper understanding of Lie symmetries and infinitesimal transformation.

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