Chapter 9

Relating the Volumes of Rectangular and Conical Frustums

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ABSTRACT

T he utility of diverse forms of knowledge is paramount for the development of any society and the continuation of its advancements. It is observed that children show increased interest in types of knowledge that they find to be applicable and relatable to their experiences. Building on an earlier presentation that discussed the volumes of various frustums, this paper delves into exploring the relationship between the volumes of Conic and Rectangular Frustums. Moreover, it demonstrates that it is feasible to estimate the volume of one frustum type based on the known volume of the other, provided certain conditions, as outlined by Olagunju (2011), are satisfied. This is accomplished through the application of validated formulas for calculating the volumes of Conical and Rectangular Frustums, $V_{CF} = \frac{1}{12}\pi h\left(D^2 + Dd + d^2\right)$ and $V_{RF} = \frac{h}{3\sqrt{5}}(D^2 + Dd + d^2)$ respectively, which were found to be related.

Keywords: Volume, Pyramids, Frustum, Rectangular, Conical, Diagonals.

BACKGROUND

Progressive scientific knowledge necessitates advancements in application to foster continuous development. This is particularly true for mathematics education, where there is a constant need to refine and improve upon existing problem-solving methods. This necessity is the driving force behind revisiting proven formulas for calculating the volumes of Rectangular-based and Circular-based Frustums and devising a formula to bridge the two.

SIGNIFICANCE OF THE STUDY

This linkage aims to enable the use of the volume of one type of frustum to estimate the volume of the other, thereby streamlining estimations for scholars and other stakeholders.

CLASSIFICATION OF PYRAMIDS AND FRUSTUMS

Olagunju (2011) characterizes Pyramids as polyhedrons with one polygonal face known as the 'base' and all other faces as triangles converging at a point referred to as the 'apex'. A Cone or a Conical Pyramid is identified as a special kind of pyramid whose base is circular, and all slant edges converge at the vertex. When a section of a Pyramid is sliced away from the apex, it results in a Truncated Pyramid, commonly termed a Frustum. Pyramids are categorized by their dimensions, with a Regular Pyramid having a base that is a regular polygon (e.g., Square-Based, Rectangular-Based) and a Right Pyramid being one where the apex is connected to the center of the base by a perpendicular line. An Arbitrary Pyramid is described as having a singular cross-sectional shape with lengths that scale linearly with its height.

BRIEF INFORMATION ON VOLUMES OF CONICAL AND RECTANGULAR FRUSTUMS

Olagunju (2011, 2016), drawing upon the work of Harris and Stocker (1998), demonstrated that the volume of a pyramid is calculated as one-third of the product of the base area and the perpendicular height. This can be succinctly expressed as:

That is Pyramidal Volume = $\frac{1}{3}$ (base-area x $\perp$ height)

Thus, the volume of the Truncated Pyramid (i.e. Pyramidal Frustum) is given as the difference between the complete pyramid and the chopped small pyramid.
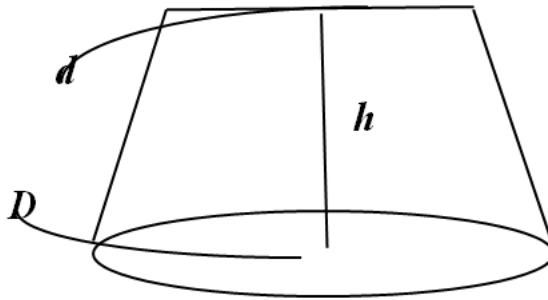
That is $V_{PF} = V_{BP} - V_{SP}$

where V_{PF} = Volume of the Pyramidal Frustum,

V_{BP} = Volume of the complete Big Pyramid

V_{SP} = Volume of the small chopped pyramid

Volume of a Conical Frustum

Consider the Conical Frustum above.

Macrae et'al (2001) led us through the method used to obtain the volume of a Conical Frustum (or simply 'a cone') popularly referred to Hero's formula, here designated as V_{CF} is given as $V_{CF} = \dfrac{1}{3}\pi h(R^2 + Rr + r^2)$

Where V_{CF} = Volume of Square-based Frustum,

       R = Radius of the large Circular base,
       r = Radius of the small Circular top,
and   h = the height of the Conical Frustum.

But Olagunju (2011) considered the said hero's formula thus:

As usual, let **D** = Diameter of circular base, **d** = Diameter of circular top,

Applying the usual fact that $R = \dfrac{1}{2}D$ and $r = \dfrac{1}{2}d$,

Then, Conical Frustum Volume
$$= \frac{1}{3}\pi h\left(\left(\frac{D}{2}\right)^2 + \frac{D}{2}\frac{d}{2} + \left(\frac{d}{2}\right)^2\right)$$

$$= \frac{1}{3}\pi h\left(\frac{D^2}{4} + \frac{Dd}{4} + \frac{d^2}{4}\right)$$

$$= \frac{1}{4}\frac{1}{3}\pi h\left(D^2 + Dd + d^2\right)$$

Hence, Conical Frustum Volume $V_{CF} = \dfrac{1}{12}\pi h\left(D^2 + Dd + d^2\right)$ (4.1)
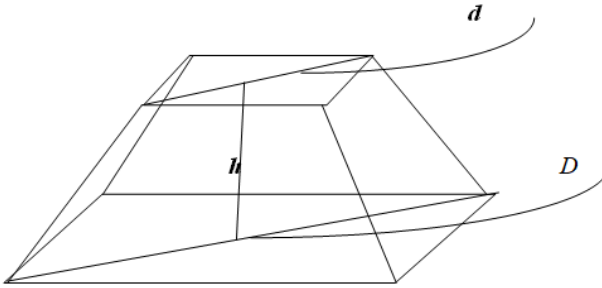
Volume of a Rectangular Frustum

Therefore, the volume of a Truncated Pyramid, also known as a Pyramidal Frustum, is determined by subtracting the volume of the smaller, removed pyramid from the volume of the complete pyramid.

In further detail, considering a Rectangular Frustum, Olagunju (2017a) applied the standard procedure of calculating the volumes of the larger and smaller pyramids. By subtracting the volume of the smaller pyramid from the larger one, he derived a tested and proven formula for the volume of a rectangular frustum. This formula takes into account:

Base-Length = **2L**, Base-breadth = **L**, Top-Length = **2l**, Top-breadth = **l**
And that **x** = *height of Small Pyramid*, **h** = *height of Frustum*,
Where **H** = *Height of Large Pyramid, (height of small pyramid + height of Frustum)*

If the base and top diagonals of the pyramidal frustum be *D* and *d* respectively.

Since its height is h, and the Volume is designated as V_{RF},

Then, by the old method, the Volume of Rectangular Frustum will be the difference between the

Volume of large Rectangular Pyramid and Volume of chopped Rectangular Pyramid.

Thus, $\quad V_{RF} = \frac{2}{3}\left[\left(\frac{D}{\sqrt{5}}\right)^2 (x+h) - \left(\frac{d}{\sqrt{5}}\right)^2 x\right]$

$\Rightarrow \qquad V_{RF} = \frac{2}{15}\left[D^2 h + x(D-d)(D+d)\right]$

Considering similarity of Δ s if the pyramids are completed, we have

$$V_{RF} = \frac{2}{15}\left[D^2 h + dh(D+d)\right]$$

Hence $\qquad V_{RF} = \frac{2}{15}\left[D^2 h + Ddh + d^2 h\right]$

So, the volume of a Rectangular Frustum is $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$ (4.2)

RELATIONSHIP BETWEEN VOLUMES OF CONICAL AND RECTANGULAR FRUSTUMS

Lemma

1.1 The Rectangular Frustum must have the Diagonals of the base and top rectangles to be equal to the Diameters of the respective base and top circles of the Conical Frustum.

1.2 The height of the Rectangular Frustum must be equal to the height of the Conical Frustum.

Relationship between Volumes of Rectangular and Conical Frustums

From equations (4.1) and (4.2) above,

The Volume of the Conical Frustum $V_{CF} = \dfrac{1}{12}\pi h\left(D^2 + Dd + d^2\right)$ (4.1)

And Volume of the Rectangular Frustum $V_{RF} = \dfrac{2h}{15}\left[D^2 + Dd + d^2\right]$ (4.2)

By Lemmas 1.1,

Height of Rectangular Frustum = Height of Conical Frustum

Also by Lemma 1.2,

Diagonal of the Top small Rectangle = Diameter of the Top small Circle and Diagonal of the Base big Rectangle = Diameter of the Base big Circle

Thus, relating equations (4.1) and (4.2),

we have $\quad \dfrac{V_{CF}}{V_{RF}} = \dfrac{\frac{1}{12}\pi h\left(D^2 + Dd + d^2\right)}{\frac{2h}{15}\left(D^2 + Dd + d^2\right)}$

$\Rightarrow \quad V_{CF} = \frac{5}{8}\pi\left[\frac{2h}{15}\left(D^2 + Dd + d^2\right)\right]$

$\Rightarrow \quad V_{CF} = \frac{5}{8}\pi\left[\text{Rectangular Frustum's Volume}\right]$ (5.1)

Equation (5.1) therefore shows that the Volume of a Conical Frustum equals the product of 5π and Rectangular Frustum's Volume divided by 8

SUMMARY

From the foregoing, we arrived at the facts that:

The Volume of the Conical Frustum $\qquad V_{CF} = \frac{1}{12}\pi h\left(D^2 + Dd + d^2\right)$ (4.1)

The Volume of the Rectangular Frustum $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$ (4.2)

Relating (4.1) and (4.2), we obtained $\qquad V_{CF} = \frac{5\pi}{8}\left[\frac{2h}{15}\left(D^2 + Dd + d^2\right)\right]$

$\Rightarrow \qquad V_{CF} = \frac{5\pi}{8}\left[\text{Rectangular Frustum's Volume}\right]$ (5.1)

We illustrate as follows, using the Diagonals (Diameters) only, irrespective of the lengths and radii.

Illustration I

Suppose that $h = 5,\ d = 4$ and $D = 8,$

Then, we have from (4.1),

$$V_{CF} = \frac{1}{12}\pi h\left(D^2 + Dd + d^2\right)$$
$$= \frac{5\pi}{12}\left((8)^2 + (8)(4) + (4)^2\right) = \frac{140\pi}{3} \qquad (6.1.1)$$

And we have from (4.2),

$$V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$$
$$= \frac{10}{15}\left[(8)^2 + (8)(4) + (4)^2\right] = \frac{224}{3} \qquad (6.1.2)$$

But $\quad \frac{140\pi}{3} = \frac{5\pi}{8}\left(\frac{224}{3}\right)$ (6.1.3)

From (6.1.3), it is clear that $V_{CF} = \dfrac{5\pi}{8} V_{RF}$.

Hence, equation (5.1) holds.

Illustration II

Suppose that $h = 7$, $d = 11$ and $D = 22$,

Then, we have from (4.1),

$$V_{CF} = \frac{1}{12}\pi h\left(D^2 + Dd + d^2\right)$$

$$= \frac{7\pi}{12}\left((22)^2 + (22)(11) + (11)^2\right) = \frac{5929\pi}{12} \qquad (6.2.1)$$

And we have from (4.2),

$$V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$$

$$= \frac{14}{15}\left[(22)^2 + (22)(11) + (11)^2\right] = \frac{11858}{15} \qquad (6.2.2)$$

But $\quad \dfrac{5929\pi}{12} = \dfrac{5\pi}{8}\left(\dfrac{11858}{15}\right) \qquad (6.2.3)$

From (6.2.3), it is clear that $V_{CF} = \dfrac{5\pi}{8} V_{RF}$.

Hence, equation (5.1) holds.

Illustration III

Suppose that $h = 0.8$, $d = 0.3$ and $D = 0.6$,

Then, we have from (4.1),

$$V_{CF} = \frac{1}{12}\pi h\left(D^2 + Dd + d^2\right)$$

$$= \frac{0.8\pi}{12}\left((0.6)^2 + (0.6)(0.3) + (0.3)^2\right) = \frac{0.126\pi}{3} \qquad (6.3.1)$$

And we have from (4.2),

$$V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$$

$$= \frac{1.6}{15}\left[(0.6)^2 + (0.6)(0.3) + (0.3)^2\right] = \frac{0.336}{5} \qquad (6.3.2)$$

But $\quad \dfrac{0.126\pi}{3} = \dfrac{5\pi}{8}\left(\dfrac{0.336}{5}\right)$ \hfill (6.3.3)

From (6.3.3), it is clear that $V_{CF} = \dfrac{5\pi}{8}V_{RF}$.

Hence, equation (5.1) holds.

Illustration IV

Suppose that $h = 14$, $d = 4$ and $D = 12$,

Then, we have from (4.1),

$$V_{CF} = \frac{1}{12}\pi h\left(D^2 + Dd + d^2\right)$$

$$= \frac{14\pi}{12}\left((12)^2 + (12)(4) + (4)^2\right) = \frac{2912\pi}{12} \qquad (6.4.1)$$

And we have from (4.2),

$$V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$$

$$= \frac{28}{15}\left[(12)^2 + (12)(4) + (4)^2\right] = \frac{5824}{15} \qquad (6.4.2)$$

But $\quad \dfrac{2912\pi}{12} = \dfrac{5\pi}{8}\left(\dfrac{5824}{15}\right)$ \hfill (6.4.3)

From (6.4.3), it is clear that $V_{CF} = \dfrac{5\pi}{8} V_{RF}$.

Hence, equation (5.1) holds.

Illustration V

Suppose that $h = 2.7$, $d = 0.6$ and $D = 2.4$,

Then, we have from (4.1),

$$
\begin{aligned}
V_{CF} &= \frac{1}{12} \pi h \left(D^2 + Dd + d^2 \right) \\
&= \frac{2.7\pi}{12} \left((2.4)^2 + (2.4)(0.6) + (0.6)^2 \right) = \frac{20.412\pi}{12}
\end{aligned}
\qquad (6.5.1)
$$

And we have from (4.2),

$$
\begin{aligned}
V_{RF} &= \frac{2h}{15} \left[D^2 + Dd + d^2 \right] \\
&= \frac{5.4}{15} \left[(2.4)^2 + (2.4)(0.6) + (0.6)^2 \right] = \frac{40.824}{15}
\end{aligned}
\qquad (6.5.2)
$$

But $\quad \dfrac{20.412\pi}{12} = \dfrac{5\pi}{8} \left(\dfrac{40.824}{15} \right)$ $\qquad (6.5.3)$

From (6.5.3), it is clear that $V_{CF} = \dfrac{5\pi}{8} V_{RF}$.

Hence, equation (5.1) holds.

Illustration VI

Suppose that $h = 10.2$, $d = 3.0$ and $D = 7.5$,

Then, we have from (4.1),

$$V_{CF} = \frac{1}{12}\pi h\left(D^2 + Dd + d^2\right)$$

$$= \frac{10.2\pi}{12}\left((7.5)^2 + (7.5)(3.0) + (3.0)^2\right) = \frac{895.05\pi}{12} \quad (6.6.1)$$

And we have from (4.2),

$$V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$$

$$= \frac{20.4}{15}\left[(7.5)^2 + (7.5)(3.0) + (3.0)^2\right] = \frac{1790.1}{15} \quad (6.6.2)$$

But $\quad \dfrac{140\pi}{3} = \dfrac{5\pi}{8}\left(\dfrac{224}{3}\right) \quad\quad\quad\quad\quad\quad (6.6.3)$

From (6.6.3), it is clear that $V_{CF} = \dfrac{5\pi}{8}V_{RF}$.

Hence, equation (5.1) holds.

PRECAUTION

However, it is critical to highlight that each model or formula comes with its own set of caveats. The accuracy of this method is contingent upon the strict adherence to the stated conditions and the correct application of the formulas. Deviations from these prerequisites might result in discrepancies between the estimated and actual volumes, thus not achieving the anticipated precision.

RECOMMENDATION

Given this established connection, it is advised that students and institutions looking to calculate the volume of a Conical Frustum can now do so more conveniently by referring to the volume of a Rectangular Frustum, assuming the latter is already known. This method offers a practical approach to volume estimation, enhancing educational tools and practical applications in fields requiring geometric computations. Nonetheless, the importance of precision in applying the conditions and

formulas cannot be overstressed, ensuring the reliability and accuracy of the volume estimations derived from this method.

CONCLUSION

Following the discussions and formulae presented, a significant linkage has been formed between the volume of a Conical Frustum and that of a Rectangular Frustum. This relationship suggests that the volume of a Conical Frustum can be approximated using the known volume of a Rectangular Frustum. Hence, the volume of one frustum type can serve as a basis for estimating the volume of the other, on the condition that the Top and Base Diagonals of the Rectangular Frustum are equivalent to the Top and Base Diameters of the Conical Frustum in question.

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